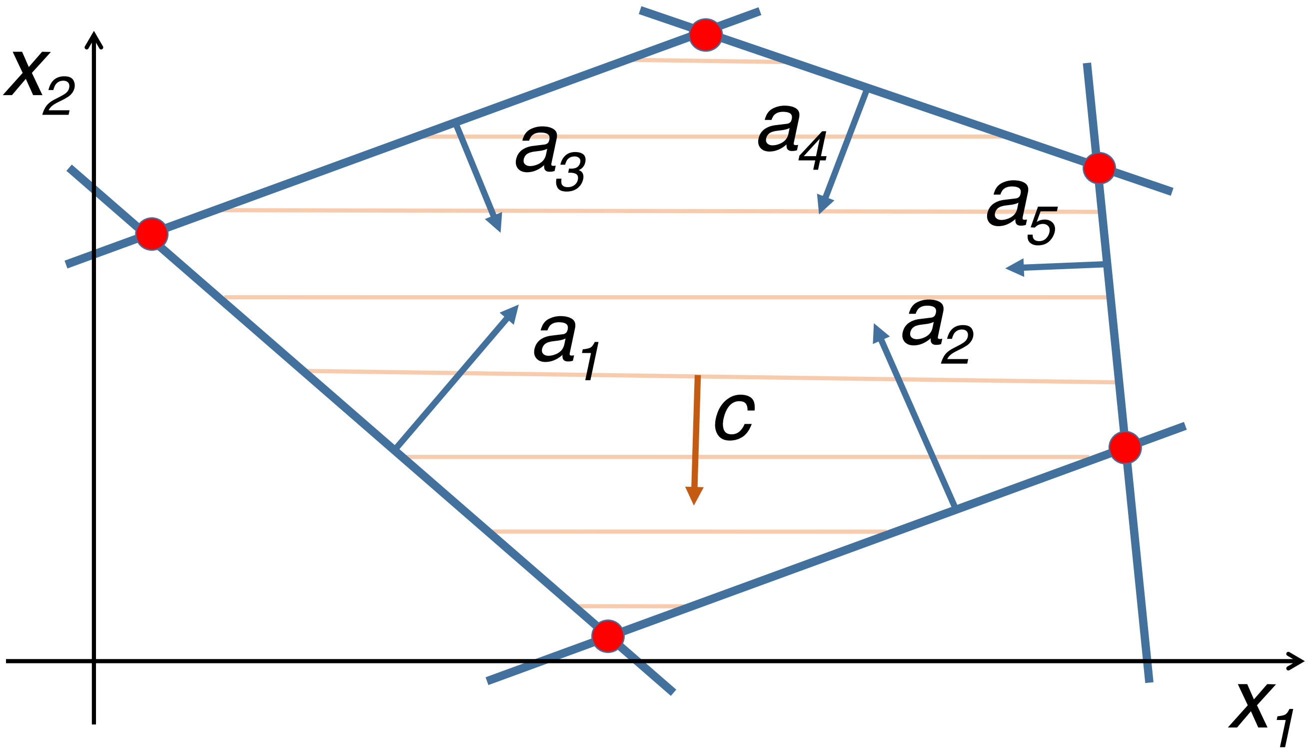
What is LP

Generally speaking, all problems with linear objective and linear equalities\inequalities constraints could be considered as Linear Programming. However, there are some widely accepted formulations.

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b
\end{aligned}
\tag{LP.Basic}
$$



for some vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$. Where the inequalities are interpreted component-wise.

Standard form

This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax = b \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Standard}
$$

Canonical form

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Canonical}
$$

Real world problems

Diet problem

Imagine, that you have to construct a diet plan from some set of products: 🍌🍰🍗🥚🐟. Each of the products has its own vector of nutrients. Thus, all the food information could be processed through the matrix W. Let also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^p} c^\top x \\
\text{s.t. } & Wx \geq r \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
$$

$c \in \mathbb{R}^p$ - cost per 100 g

$$
\min_{x \in \mathbb{R}^p} c^\top x
$$

$$
Wx \geq r
$$

How to retrieve LP

Basic transformations

Inequality to equality by increasing the dimension of the problem by m.

$$
Ax \leq b \leftrightarrow \begin{cases} Ax + z = b \\ z \geq 0 \end{cases}
$$

unsigned variables to nonnegative variables.

$$
x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}
$$

Chebyshev approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_i |a_i^\top x - b_i|
$$

$$
\begin{split}
&\min_{t \in \mathbb{R}, x \in \mathbb{R}^n} t \\
\text{s.t. } & a_i^\top x - b_i \leq t, \; i = 1, \dots, n \\
& - a_i^\top x + b_i \leq t, \; i = 1, \dots, n
\end{split}
$$

l_1 approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^\top x - b_i|
$$

$$
\begin{split}
&\min_{t \in \mathbb{R}^n, x \in \mathbb{R}^n} \mathbf{1}^\top t \\
\text{s.t. } & a_i^\top x - b_i \leq t_i, \; i = 1, \dots, n \\
& - a_i^\top x + b_i \leq t_i, \; i = 1, \dots, n
\end{split}
$$

Idea of simplex algorithm

- The Simplex Algorithm walks along the edges of the polytope, at every corner choosing the edge that decreases $c^\top x$ most
- This either terminates at a corner, or leads to an unconstrained edge ($-\infty$ optimum)

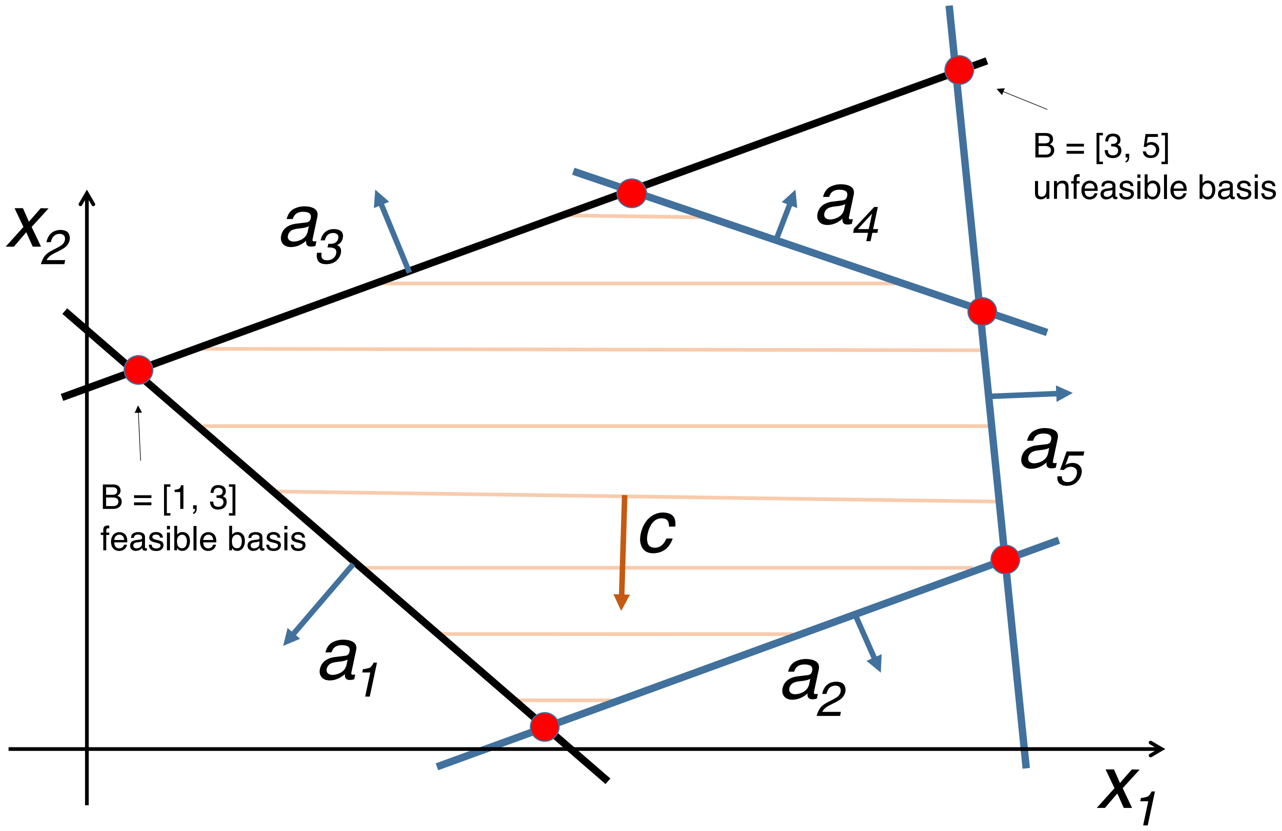
We will illustrate simplex algorithm for the simple inequality form of LP:

$$
\begin{split}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b
\end{split}
\tag{LP.Inequality}
$$

Definition: a **basis** B is a subset of n (integer) numbers between 1 and m, so that $\text{rank} A_B = n$. Note, that we can associate submatrix A_B and corresponding right-hand side b_B with the basis B. Also, we can derive a point of intersection of all these hyperplanes from basis: $x_B = A^{-1}_B b_B$.

If $Ax_B \leq b$, then basis B is **feasible**.

A basis B is optimal if x_B is an optimum of the LP.Inequality.



Since we have a basis, we can decompose our objective vector c in this basis and find the scalar coefficients λ_B:

$$\lambda^\top_B A_B = c^\top \leftrightarrow \lambda^\top_B = c^\top A^{-1}_B$$

Main lemma

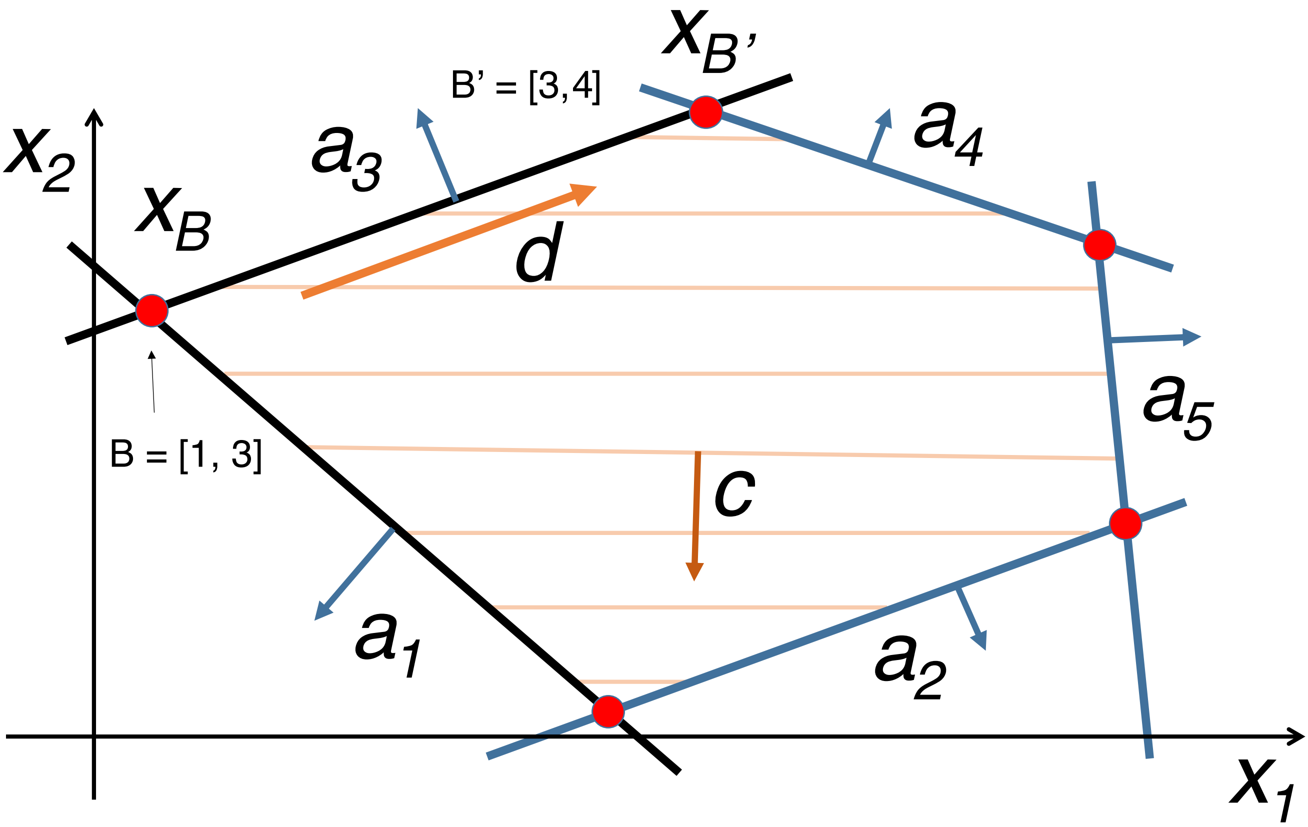
If all components of λ_B are non-positive and B is feasible, then B is optimal.

Proof:

$$
\begin{aligned}
\exists x^* : Ax^* \le b, c^\top x^* < c^\top x_B \\
A_B x^* \le b_B \\
\lambda_B^\top A_B x^* \ge \lambda_B^\top b_B \\
c^\top x^* \ge \lambda_B^\top A_B x_B \\
c^\top x^* \ge c^\top x_B
\end{aligned}
$$

Changing basis

Suppose, some of the coefficients of λ_B are positive. Then we need to go through the edge of the polytope to the new vertex (i.e., switch the basis)



$$
x_{B'} = x_B + \mu d = A_{B'}^{-1} b_{B'}
$$

Finding an initial basic feasible solution

Let us consider $\mathrm{LP.Canonical}$.

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax = b \\
& x_i \ge 0, \; i = 1, \dots, n
\end{aligned}
$$

The proposed algorithm requires an initial basic feasible solution and corresponding basis. To compute this solution and basis, we start by multiplying by -1 any row i of $Ax = b$ such that $b_i < 0$. This ensures that $b \geq 0$. We then introduce artificial variables $z \in \mathbb{R}^m$ and consider the following LP:

$$
\begin{aligned}
& \min_{x \in \mathbb{R}^n, z \in \mathbb{R}^m} 1^\top z \\
\text{s.t. } & Ax + Iz = b \\
& x_i, z_j \geq 0, \; i = 1, \dots, n \; j = 1, \dots, m
\end{aligned}
\tag{LP.Phase 1}
$$

which can be written in canonical form $\min\{\tilde{c}^\top \tilde{x} \mid \tilde{A}\tilde{x} = \tilde{b}, \tilde{x} \geq 0\}$ by setting

$$
\tilde{x} = \begin{bmatrix} x \\ z \end{bmatrix}, \quad \tilde{A} = [A \; I], \quad \tilde{b} = b, \quad \tilde{c} = \begin{bmatrix} 0_n \\ 1_m \end{bmatrix}
$$

An initial basis for $\text{LP.Phase } 1$ is $\tilde{A}_B = I, \tilde{A}_N = A$ with corresponding basic feasible solution $\tilde{x}_N = 0, \tilde{x}_B = \tilde{A}_B^{-1}\tilde{b} = \tilde{b} \geq 0$. We can therefore run the simplex method on $\text{LP.Phase } 1$, which will converge to an optimum $\tilde{x}^*$. $\tilde{x} = (\tilde{x}_N \; \tilde{x}_B)$. There are several possible outcomes:

- $$\tilde{c}^\top \tilde{x} > 0$$

  . Original primal is infeasible.

- $$\tilde{c}^\top \tilde{x} = 0 \to 1^\top z^* = 0$$

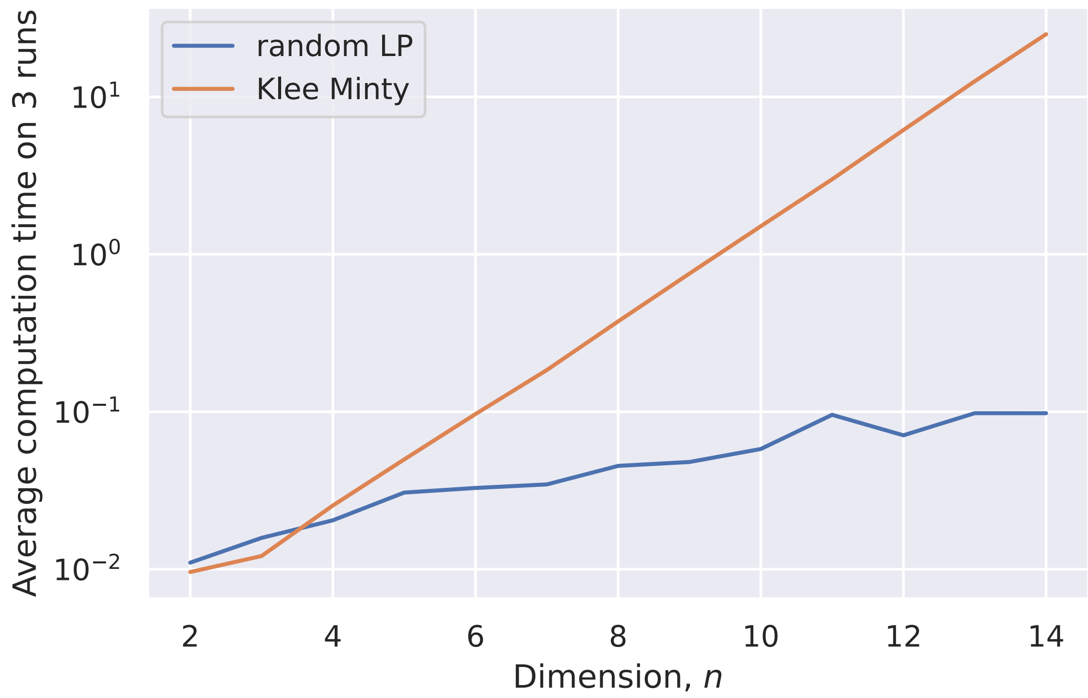
  . The obtained solution is a start point for the original problem (probably with slight modification).

Convergence

Klee Minty example

In the following problem simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$
\begin{aligned}
& \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + x_n \\
\text{s.t. } & x_1 \leq 5 \\
& 4x_1 + x_2 \leq 25 \\
& 8x_1 + 4x_2 + x_3 \leq 125 \\
& \ldots \\
& 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \ldots + x_n \leq 5^n \qquad x \geq 0
\end{aligned}
$$



Strong duality

There are four possibilities:

- Both the primal and the dual are infeasible.
- The primal is infeasible and the dual is unbounded.
- The primal is unbounded and the dual is infeasible.
- Both the primal and the dual are feasible and their optimal values are equal.

Summary

- A wide variety of applications could be formulated as the linear programming.
- Simplex algorithm is simple, but could work exponentially long.
- Khachiyan's ellipsoid method is the first to be proved running at polynomial complexity for LPs. However, it is usually slower than simplex in real problems.
- Interior point methods are the last word in this area. However, good implementations of simplex-based methods and interior point methods are similar for routine applications of linear programming.

Code

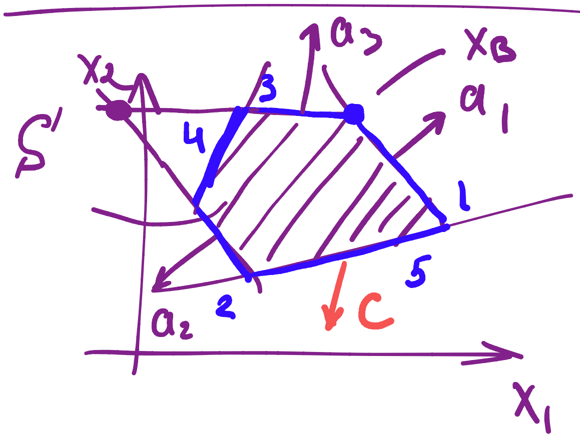
Open in Colab

Materials

- Linear Programming. in V. Lempitsky optimization course.
- Simplex method. in V. Lempitsky optimization course.
- Overview of different LP solvers
- TED talks watching optimization
- Overview of ellipsoid method
- Comprehensive overview of linear programming
- Converting LP to a standard form

Simplex algorithm

$$\min c^T x \qquad A \in \mathbb{R}^{m \times n}$$
$$x \in \mathbb{R}^n$$
$$Ax \le b \qquad A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}$$



B_{13} - оптимальный допустимый

$$a_1^T x \le b_1$$
$$\vdots$$
$$a_m^T x \le b_m$$

m огран.

naive:
перебрать все угловые точки и выбрать лучшую.

Def. Базис $B = \{i, j\}$ — мн-во индексов

$$A_B x_B = b_B$$

$B = \{1, 3\}$ ← n индексов

$$A_B = \begin{bmatrix} a_i^T \\ a_j^T \end{bmatrix} \quad b_B = \begin{pmatrix} b_i \\ b_j \end{pmatrix}$$

$$x_B = A_B^{-1} b_B$$

Def. Базис B - допустимый, если: $A x_B \le b$

Def. Базис B - оптимальный $\quad c^T x_B \le c^T x \quad \forall x \in S$

Th. Давайте разложим вектор целевой функции c по выбранному базису: $\Rightarrow$

$$c = A_B^T \cdot \lambda_B = \sum_{i=1}^{n} \lambda_i \cdot a_{i_B}$$

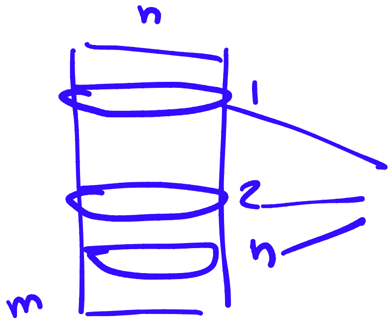
← коэффициенты разл.

из базиса

Если $\boxed{\lambda_B \le 0}$ и B - допустимый, то B - оптимальный.

пусть $\exists x^*$, что $\underbrace{Ax^* \leq b}_{\downarrow\ \text{допустимость}}$; $\underbrace{c^T x^* < c^T x_B}$

$\lambda \leq 0$



$A_B x^* \leq b_B \quad \Big| \; \lambda_B^T \cdot$

$\boxed{c = A_B^T \lambda_B}$

$\underbrace{\lambda_B^T A_B}_{} x^* \geq \lambda_B^T b_B$

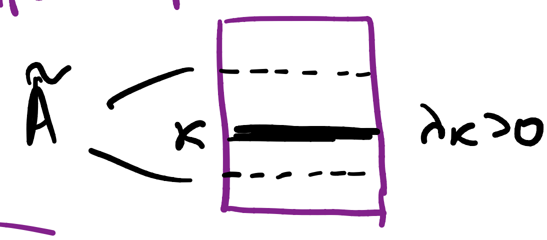
$c^T = \lambda_B^T A_B$

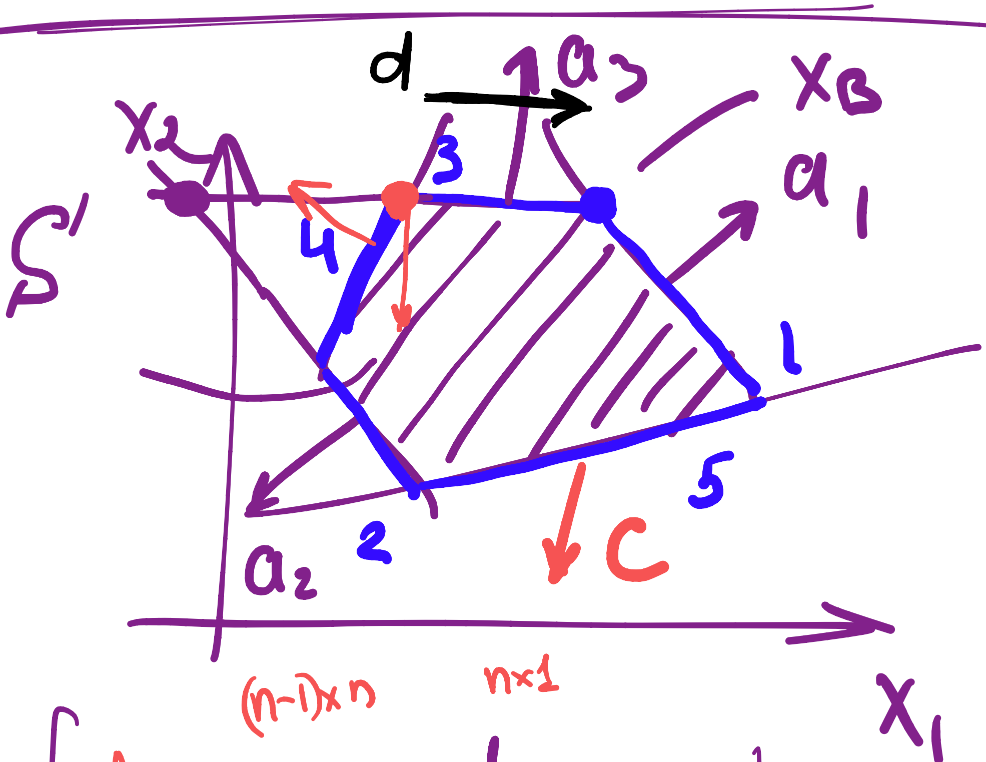
$c^T x^* \geq \underbrace{\lambda_B^T A_B}_{c^T} x_B$

$b_B = A_B x_B$

$c^T x^* \geq c^T x_B$ — Противоречие.

ЗАМЕНА БАЗИСА.





$B = \{3, 4\}$

неоптимальный

$\lambda_B = A_B^{-T} c$

$c = A_B^T \lambda_B$

Пусть $\exists k: \lambda_k > 0$

$x_{B'} = x_B + \mu \cdot d$

$\cancel{\tilde{A} x_{B'} = \tilde{A} x_B + \mu \tilde{A} d}$

$\cancel{\tilde{b}_{(n-1)}} \qquad \cancel{\tilde{b}}$

$(n-1) \times n \qquad n \times 1$

$$\begin{cases} \underset{n-1}{\;} A_{B \setminus \{B_k\}} \cdot d = 0 \\ \underset{1}{\;} a_{B_k}^T \cdot d = -1 \end{cases}$$

условия для d

$$\mu_j = \frac{b_j - a_j^{T} x_B}{a_j^{T} d}$$

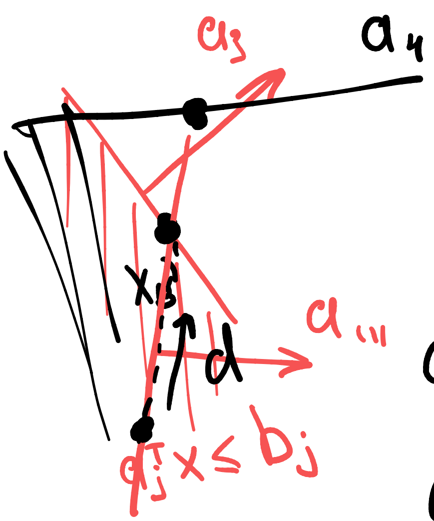
$j \notin B$ Считаем для всех $j \notin B$ $(m-n)$ штук и выбираем

$$t = \underset{j}{\operatorname{argmin}} \{ \mu_j \mid \mu_j > 0 \}$$

Новый базис:

$$B' = B \setminus \{B_k\} \cup \{t\}$$

$$x_{B'} = A_{B'}^{-1} \cdot b_{B'}$$



$$a_j^T \cdot x_B' = b_j$$

$$a_j^T (x_B + \mu d) = b_j$$

$$x_B' = x_B + \mu d$$

$$\mu a_j^T d = b_j - a_j^T x_B$$

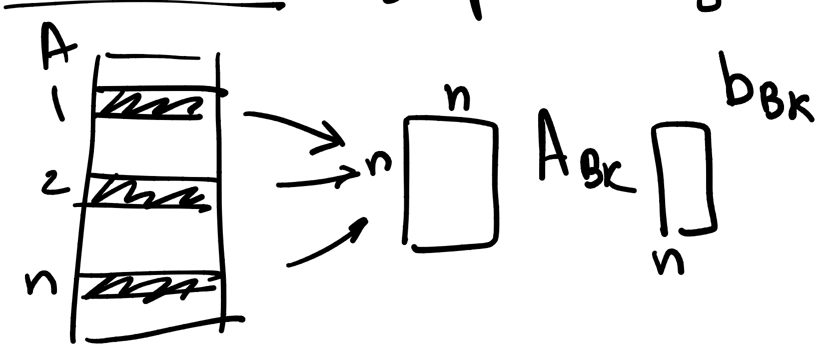
$$c^T x \to \min_{x \in \mathbb{R}^n}$$

$$\text{s.t. } Ax \le b \qquad m > n$$

Алгоритм

(?) Шаг 1 Выбрать допустимый базис B_k $\Rightarrow$ $A_{B_k} x_k = b_{B_k}$

$$x_k = A_{B_k}^{-1} b_{B_k}$$



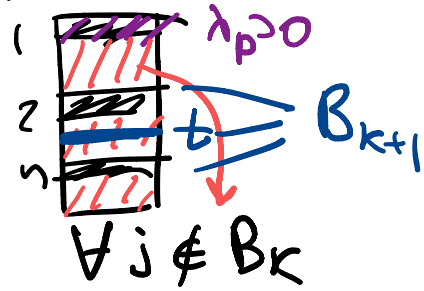
Шаг 2 Разложить c в выбранный базис $c = A_{B_k}^T \cdot \lambda_{B_k}$

Шаг 3 Проверить оптимальность $\lambda_{B_k} \le 0$

3.1 Если $\lambda_{B_k} \le 0 \Rightarrow x_k$ – решение

3.2 Если какая-то компонента $\lambda_p > 0$, то заменяем базис:

$$x_{k+1} = x_k + \mu_k d_k$$



$$d_k = \begin{cases} A_{B_k \setminus \{B_p\}} \cdot d = 0 \\ a_p^T d = -1 \end{cases}$$

$$\mu_j = \frac{b_j - a_j^T x_k}{a_j^T d_k}$$

$$t = \underset{j}{\operatorname{argmin}} \{\mu_j \mid \mu_j > 0\}$$

(если $\mu_j < 0$, то задача не огран.)

$$\Rightarrow B_{k+1} = B_k \setminus \{B_p\} \cup \{t\} \qquad x_{k+1} = A^{-1}_{B_{k+1}} b_{B_{k+1}}$$

Вернитесь на шаг 2

как минимум на $\mu_k c^T d_k$ уменьшаем

$$\min_{x\in\mathbb{R}^n} c^T x \quad Ax \le b \iff \boxed{\begin{aligned} &\min c^T(y-z) \\ &\text{s.t. } y\in\mathbb{R}^n,\ z\in\mathbb{R}^n \\ &Ay - Az \le b \\ &y \ge 0,\ z \ge 0 \end{aligned}}$$

2n перем.

m+2n огран.

Phase 2

$x = y - z$

$\Leftarrow \quad x_i = y_i - z_i$

$\Rightarrow \quad y_i = \max(x_i, 0)$

$z_i = \max(-x_i, 0)$

$$\underset{x}{\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}} = \underset{y}{\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}} - \underset{z}{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}$$

Phase 1

$$\boxed{\begin{aligned} &\min \sum_{i=1}^{m} \xi_i \\ &\text{s.t. } \xi\in\mathbb{R}^m,\ z\in\mathbb{R}^n,\ y\in\mathbb{R}^n \\ &Ay - Az \le b + \xi \\ &y \ge 0,\ z \ge 0,\ \xi \ge 0 \end{aligned}}$$

m+2n переменных

2m+2n ограничений

<u>Утв. 1</u> Если Phase-2 (Main LP) имеет допустимое решение, то оптимальное значение Phase 1 $=0$

<u>Утв. 2</u> Если оптим. значение Phase 1 $=0$, то решение Ph. 1 будет допустимым базисом для Ph. 2.

$\underbrace{\xi_i = 0}_{m \text{ штук}}$, y, z $\rightarrow$ Ph. 2

m+2n ограничений нер-в

Phase 1 initialization

$$z = 0 \qquad y = 0 \qquad \xi_i = \max(0, -b_i)$$

n акт. огр. $\qquad$ n акт. огр. $\qquad$ m акт. огр.

1953

Данциг

Канторович

30-ые ?