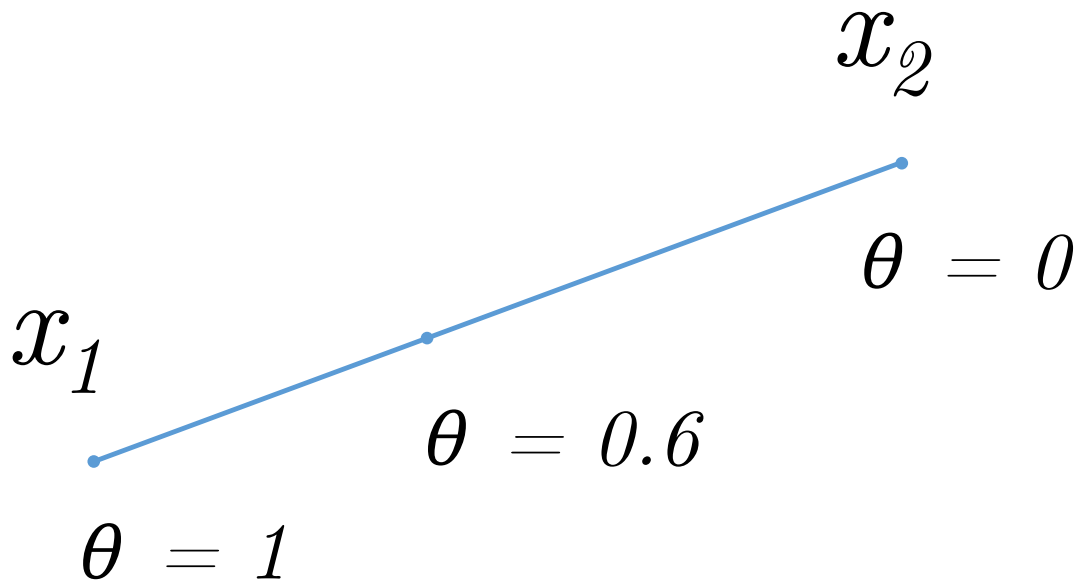


Convex set

Line segment

Suppose x_1, x_2 are two points in \mathbb{R}^n . Then the line segment between them is defined as follows:

$$x = \theta x_1 + (1 - \theta)x_2, \theta \in [0, 1]$$



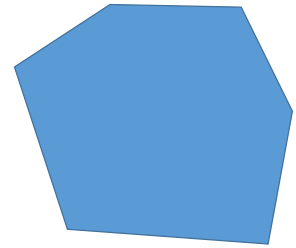
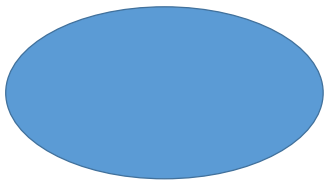
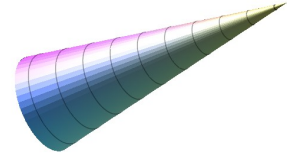
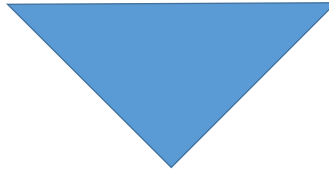
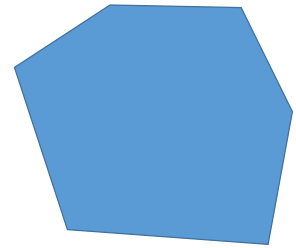
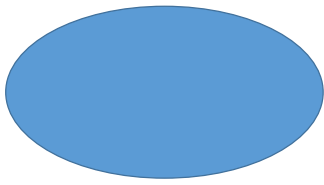
Convex set

The set S is called **convex** if for any x_1, x_2 from S the line segment between them also lies in S , i.e.

$$\forall \theta \in [0, 1], \forall x_1, x_2 \in S : \\ \theta x_1 + (1 - \theta)x_2 \in S$$

Examples:

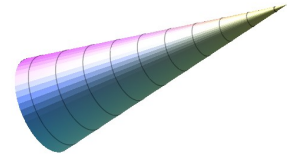
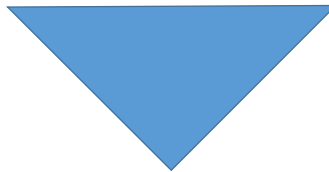
- Any affine set
- Ray
- Line segment



BRO

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Related definitions

Convex combination

Let $x_1, x_2, \dots, x_k \in S$, then the point $\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$ is called the convex combination of points x_1, x_2, \dots, x_k if $\sum_{i=1}^k \theta_i = 1$, $\theta_i \geq 0$

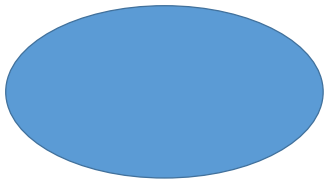
Convex hull

The set of all convex combinations of points from S is called the convex hull of the set S .

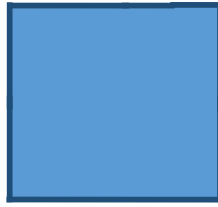
$$\mathbf{conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \mid x_i \in S, \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0 \right\}$$

- The set $\mathbf{conv}(S)$ is the smallest convex set containing S .
- The set S is convex if and only if $S = \mathbf{conv}(S)$.

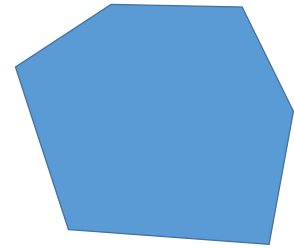
Examples:



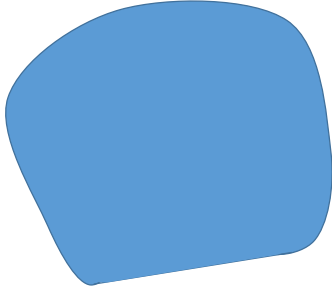
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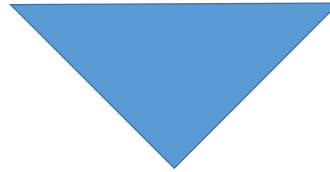
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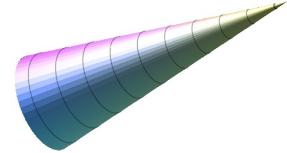
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Finding convexity

In practice it is very important to understand whether a specific set is convex or not. Two approaches are used for this depending on the context.

- By definition.
- Show that S is derived from simple convex sets using operations that preserve convexity.

By definition

$$x_1, x_2 \in S, 0 \leq \theta \leq 1 \rightarrow \theta x_1 + (1 - \theta)x_2 \in S$$

Preserving convexity

The linear combination of convex sets is convex

Let there be 2 convex sets S_x, S_y , let the set $S = \{s \mid s = c_1x + c_2y, x \in S_x, y \in S_y, c_1, c_2 \in \mathbb{R}\}$

Take two points from S : $s_1 = c_1x_1 + c_2y_1, s_2 = c_1x_2 + c_2y_2$ and prove that the segment between them $\theta s_1 + (1 - \theta)s_2, \theta \in [0, 1]$ also belongs to S

$$\theta s_1 + (1 - \theta)s_2$$

$$\theta(c_1x_1 + c_2y_1) + (1 - \theta)(c_1x_2 + c_2y_2)$$

$$c_1(\theta x_1 + (1 - \theta)x_2) + c_2(\theta y_1 + (1 - \theta)y_2)$$

$$c_1x + c_2y \in S$$

The intersection of any (!) number of convex sets is convex

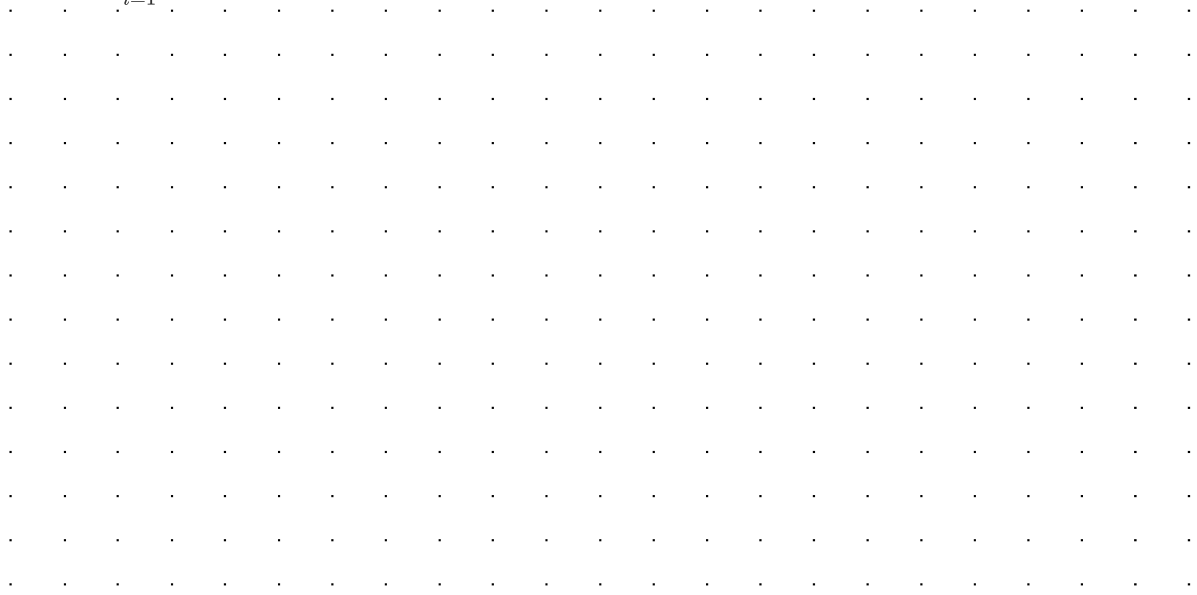
If the desired intersection is empty or contains one point, the property is proved by definition. Otherwise, take 2 points and a segment between them. These points must lie in all intersecting sets, and since they are all convex, the segment between them lies in all sets and, therefore, in their intersection.

Example 3

Let $x \in \mathbb{R}$ is a random variable with a given probability distribution of $\mathbb{P}(x = a_i) = p_i$, where $i = 1, \dots, n$, and $a_1 < \dots < a_n$. It is said that the probability vector of outcomes of $p \in \mathbb{R}^n$ belongs to the probabilistic simplex, i.e.

$P = \{p \mid \mathbf{1}^T p = 1, p \succeq 0\} = \{p \mid p_1 + \dots + p_n = 1, p_i \geq 0\}$. Determine if the following sets of p are convex: 1. $\alpha < \mathbb{E}f(x) < \beta$, where $\mathbb{E}f(x)$ stands for expected value of $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, i.e.

$$\mathbb{E}f(x) = \sum_{i=1}^n p_i f(a_i) \quad 1. \quad \mathbb{E}x^2 \leq \alpha \quad 1. \quad \forall x \leq \alpha$$



Convex function

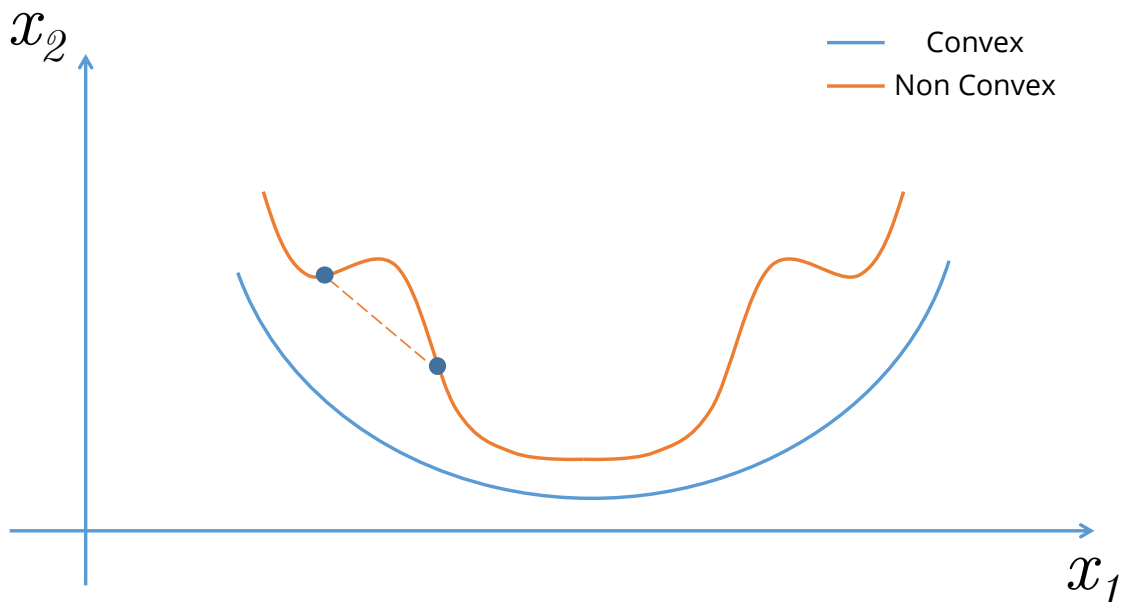
Convex function

The function $f(x)$, which is defined on the convex set $S \subseteq \mathbb{R}^n$, is called **convex** S , if:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$.

If above inequality holds as strict inequality $x_1 \neq x_2$ and $0 < \lambda < 1$, then function is called strictly convex S



Examples

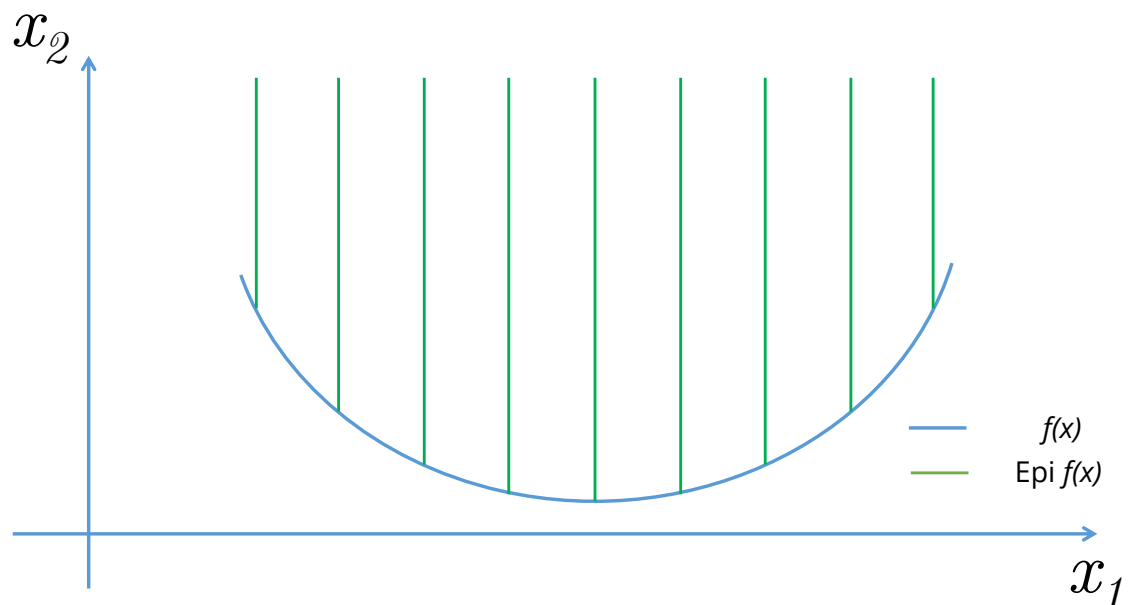
- $f(x) = x^p, p > 1, S = \mathbb{R}_+$
- $f(x) = \|x\|^p, p > 1, S = \mathbb{R}$
- $f(x) = e^{cx}, c \in \mathbb{R}, S = \mathbb{R}$
- $f(x) = -\ln x, S = \mathbb{R}_{++}$
- $f(x) = x \ln x, S = \mathbb{R}_{++}$
- The sum of the largest k coordinates $f(x) = x_{(1)} + \dots + x_{(k)}, S = \mathbb{R}^n$
- $f(X) = \lambda_{\max}(X), X = X^T$
- $f(X) = -\log \det X, S = S_{++}^n$

Epigraph

For the function $f(x)$, defined on $S \subseteq \mathbb{R}^n$, the following set:

$$\text{epi } f = \{[x, \mu] \in S \times \mathbb{R} : f(x) \leq \mu\}$$

is called **epigraph** of the function $f(x)$

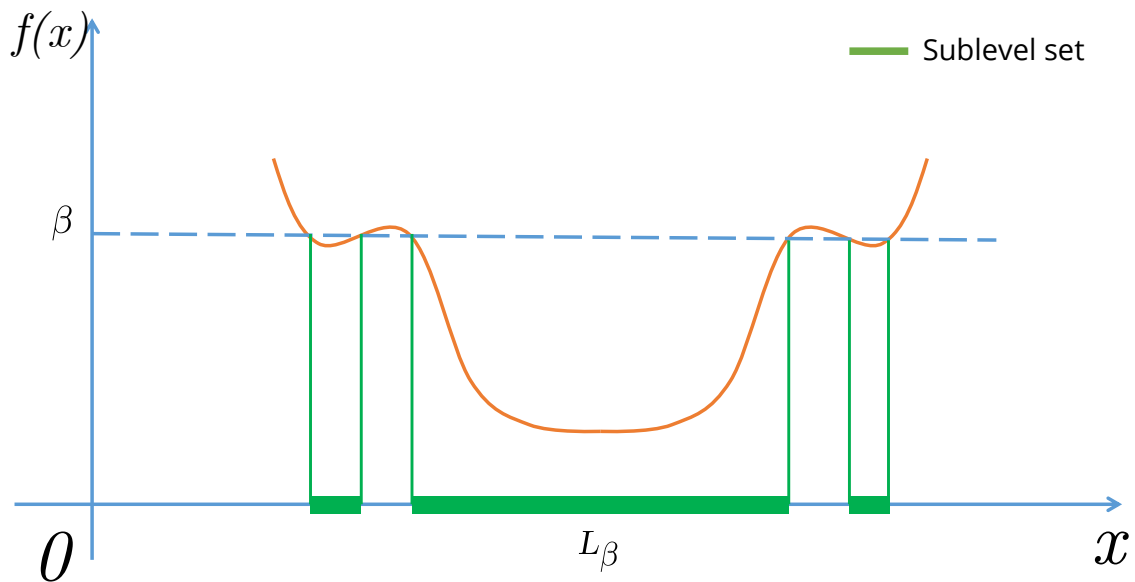


Sublevel set

For the function $f(x)$, defined on $S \subseteq \mathbb{R}^n$, the following set:

$$\mathcal{L}_\beta = \{x \in S : f(x) \leq \beta\}$$

is called **sublevel set** or Lebesgue set of the function $f(x)$



Criteria of convexity

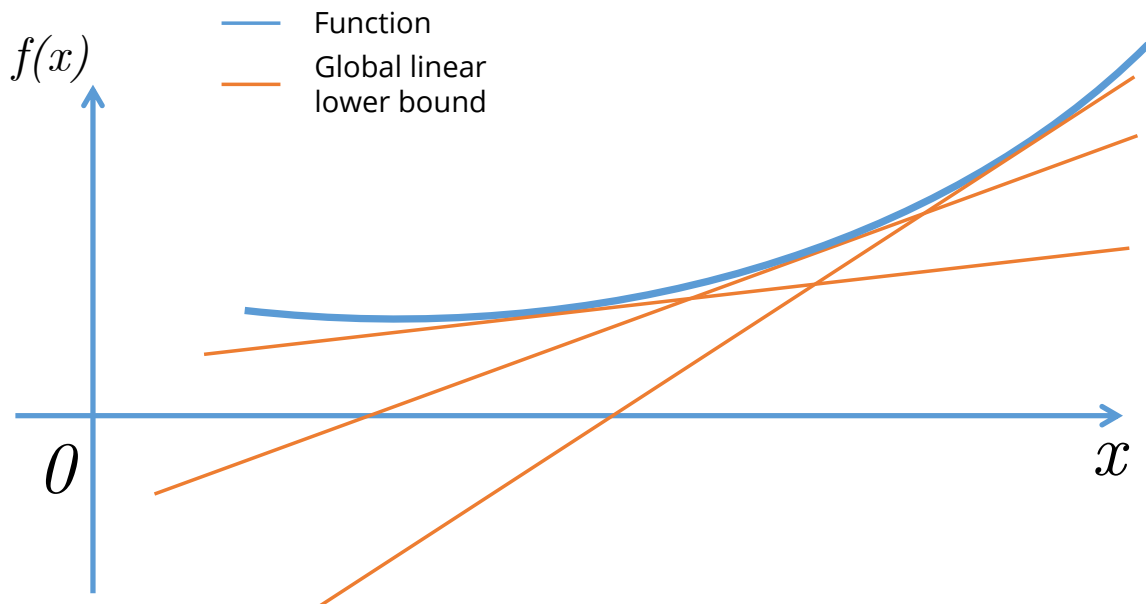
First order differential criterion of convexity

The differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x, y \in S$:

$$f(y) \geq f(x) + \nabla f^T(x)(y - x)$$

Let $y = x + \Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x$$



Second order differential criterion of convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is convex if and only if $\forall x \in \text{int}(S) \neq \emptyset$:

$$\nabla^2 f(x) \succeq 0$$

In other words, $\forall y \in \mathbb{R}^n$:

$$\langle y, \nabla^2 f(x)y \rangle \geq 0$$

Connection with epigraph

The function is convex if and only if its epigraph is convex set.

Connection with sublevel set

If $f(x)$ is a convex function defined on the convex set $S \subseteq \mathbb{R}^n$, then for any β sublevel set \mathcal{L}_β is convex.

The function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is closed if and only if for any β sublevel set \mathcal{L}_β is closed.

Reduction to a line

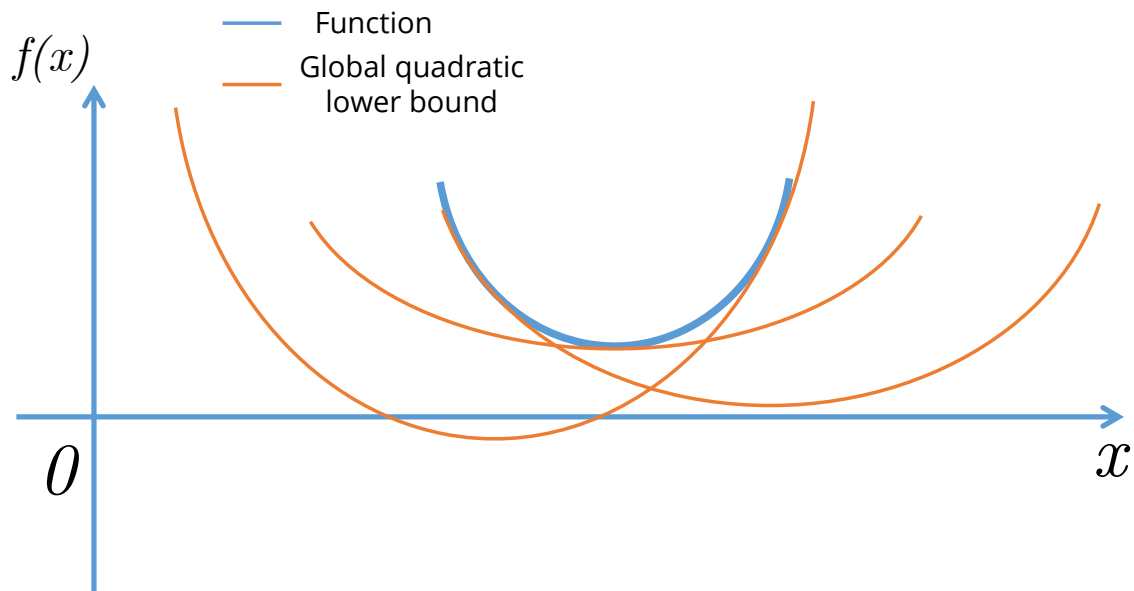
$f : S \rightarrow \mathbb{R}$ is convex if and only if S is convex set and the function $g(t) = f(x + tv)$ defined on $\{t \mid x + tv \in S\}$ is convex for any $x \in S, v \in \mathbb{R}^n$, which allows to check convexity of the scalar function in order to establish convexity of the vector function.

Strong convexity

$f(x)$, defined on the convex set $S \subseteq \mathbb{R}^n$, is called μ -strongly convex (strongly convex) on S , if:

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) - \mu\lambda(1 - \lambda)\|x_1 - x_2\|^2$$

for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$ for some $\mu > 0$.



Criteria of strong convexity

First order differential criterion of strong convexity

Differentiable $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ μ -strongly convex if and only if $\forall x, y \in S$:

$$f(y) \geq f(x) + \nabla f^T(x)(y - x) + \frac{\mu}{2}\|y - x\|^2$$

Let $y = x + \Delta x$, then the criterion will become more tractable:

$$f(x + \Delta x) \geq f(x) + \nabla f^T(x)\Delta x + \frac{\mu}{2}\|\Delta x\|^2$$

Second order differential criterion of strong convexity

Twice differentiable function $f(x)$ defined on the convex set $S \subseteq \mathbb{R}^n$ is called μ -strongly convex if and only if $\forall x \in \text{int}(S) \neq \emptyset$:

$$\nabla^2 f(x) \succeq \mu I$$

In other words:

$$\langle y, \nabla^2 f(x)y \rangle \geq \mu\|y\|^2$$

Facts

- $f(x)$ is called (strictly) concave, if the function $-f(x)$ - (strictly) convex.
- Jensen's inequality for the convex functions:

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \leq \sum_{i=1}^n \alpha_i f(x_i)$$

for $\alpha_i \geq 0$; $\sum_{i=1}^n \alpha_i = 1$ (probability simplex)

For the infinite dimension case:

$$f\left(\int_S xp(x)dx\right) \leq \int_S f(x)p(x)dx$$

If the integrals exist and $p(x) \geq 0$, $\int_S p(x)dx = 1$

- If the function $f(x)$ and the set S are convex, then any local minimum $x^* = \arg \min_{x \in S} f(x)$ will be the global one. Strong convexity guarantees the uniqueness of the solution.

Operations that preserve convexity

- Non-negative sum of the convex functions: $\alpha f(x) + \beta g(x)$, ($\alpha \geq 0, \beta \geq 0$)
- Composition with affine function $f(Ax + b)$ is convex, if $f(x)$ is convex
- Pointwise maximum (supremum): If $f_1(x), \dots, f_m(x)$ are convex, then $f(x) = \max\{f_1(x), \dots, f_m(x)\}$ is convex
- If $f(x, y)$ is convex on x for any $y \in Y$: $g(x) = \sup_{y \in Y} f(x, y)$ is convex
- If $f(x)$ is convex on S , then $g(x, t) = tf(x/t)$ - is convex with $x/t \in S, t > 0$
- Let $f_1 : S_1 \rightarrow \mathbb{R}$ and $f_2 : S_2 \rightarrow \mathbb{R}$, where $\text{range}(f_1) \subseteq S_2$. If f_1 and f_2 are convex, and f_2 is increasing, then $f_2 \circ f_1$ is convex on S_1

Other forms of convexity

- Log-convex: $\log f$ is convex; Log convexity implies convexity.
- Log-concavity: $\log f$ concave; **not** closed under addition!
- Exponentially convex: $[f(x_i + x_j)] \succeq 0$, for x_1, \dots, x_n
- Operator convex: $f(\lambda X + (1 - \lambda)Y) \preceq \lambda f(X) + (1 - \lambda)f(Y)$
- Quasiconvex: $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$
- Pseudoconvex: $\langle \nabla f(y), x - y \rangle \geq 0 \rightarrow f(x) \geq f(y)$

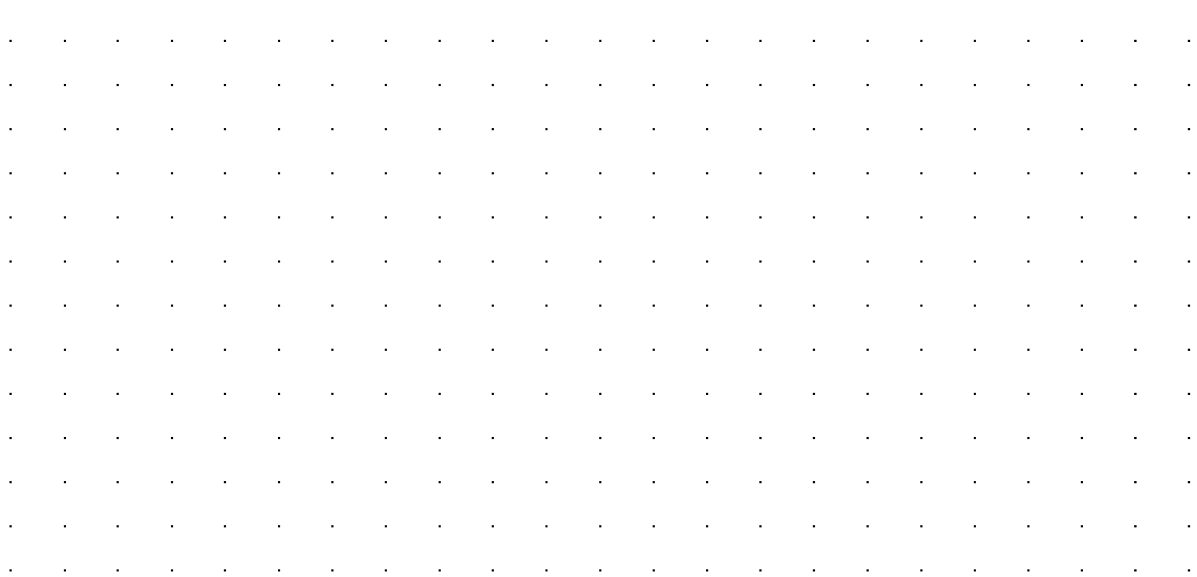
- Discrete convexity: $f : \mathbb{Z}^n \rightarrow \mathbb{Z}$; "convexity + matroid theory."

References

- [Steven Boyd lectures](#)
- [Suvrit Sra lectures](#)
- [Martin Jaggi lectures](#)

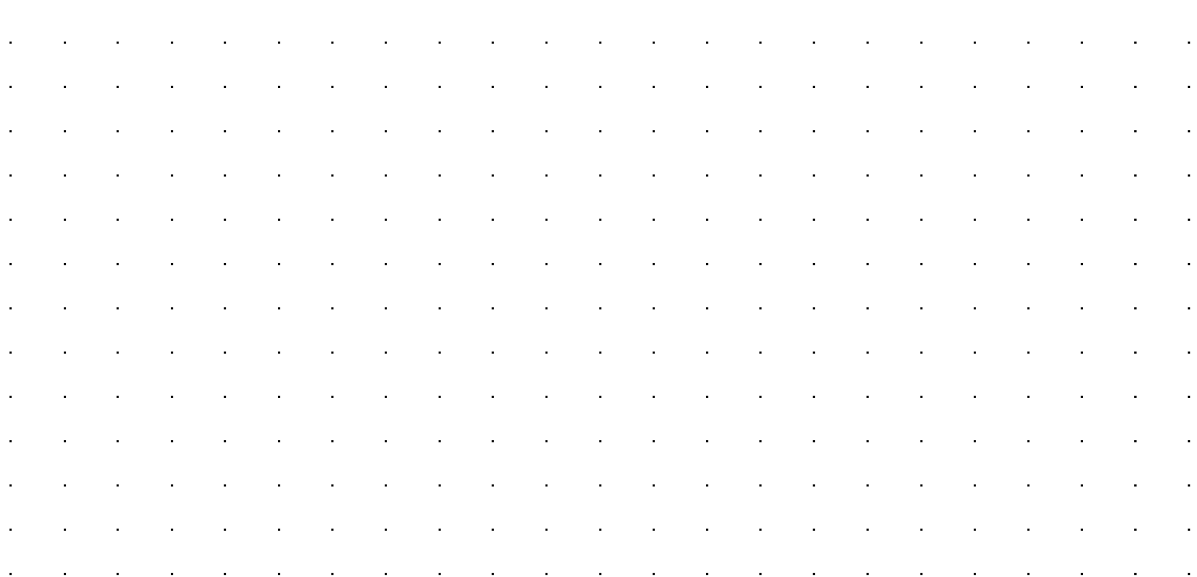
Example 4

Show, that $f(x) = c^\top x + b$ is convex and concave.



Example 5

Show, that $f(x) = x^\top Ax$, where $A \succeq 0$ - is convex on \mathbb{R}^n .



Example 6

